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A solution of the abstract Dirichlet problem for Baire-one functions ☆

Jiří Spurný, Ondřej F.K. Kalenda *

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

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Abstract

Let X be a compact convex subset of a locally convex space. We show that any bounded Baire-one function defined on $\operatorname{ext} X$ can be extended to an affine Baire-one function on X if and only if X is a Choquet simplex and $\operatorname{ext} X$ satisfies a certain topological property.

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1. Introduction

The abstract Dirichlet problem is a question of the following type. Let X be a compact convex subset of a locally convex space and f be a function defined on $\operatorname{ext} X$, the set of all extreme points of X. Can f be extended to an affine function on X that shares given properties with f?

A classical theorem of Bauer (see e.g. [1, Theorems II.4.1 and II.4.3] or [3, Satz 2]) says that *any bounded continuous function on* $\operatorname{ext} X$ *can be extended to a continuous affine function on X if and only if X is a Choquet simplex and* $\operatorname{ext} X$ *is closed in X.*

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* Corresponding author.
E-mail addresses: spurny@karlin.mff.cuni.cz (J. Spurný), kalenda@karlin.mff.cuni.cz (O.F.K. Kalenda).

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In the present paper, we give a complete solution of the analogous question for Baire-one functions (a function is *Baire-one* if it is a pointwise limit of a sequence of continuous functions). Let us briefly recall the history of this question.

It has been known for a long time that any bounded Baire-one function on $\operatorname{ext} X$ can be extended to an affine Baire-one function on X provided X is a Choquet simplex and $\operatorname{ext} X$ is F_σ in X (see e.g. [15, Theorem 37]). It was conjectured in [9] that the converse holds as well. The first author proved in [17, Theorem 2] that the converse is true within metrizable simplices (even in a more general context of simplicial function spaces). Moreover, he provided an example witnessing that outside metrizable spaces the converse is not true [17, Example 3].

Inspired by this example the second author suggested in [10] another conjecture:

Any bounded Baire-one function on $\operatorname{ext} X$ *can be extended to an affine Baire-one function on* X *if and only if* X *is a Choquet simplex and* $\operatorname{ext} X$ *is a Lindelöf* H*-set.*

H-sets are defined in [13, §12, II], where their basic properties are described. Let us recall some equivalent definitions. A subset A of a topological space X is an H-set if for any nonempty $B \subset X$ there is a nonempty relatively open $U \subset B$, such that either $U \subset A$ or $U \cap A = \emptyset$. It is clear that H-sets form an algebra containing all open sets. Further, A is an H-set in X if and only if A is the union of a scattered family of sets of the form $F \cap G$ with F closed and G open. (Recall that a family $\mathcal{U}$ of subsets of a topological space is scattered if it is disjoint and for each nonempty $\mathcal{V} \subset \mathcal{U}$ there is some $V \in \mathcal{V}$ relatively open in $\bigcup \mathcal{V}$.)

It follows from the already quoted result of [17] that the conjecture is valid within metrizable simplices. (Note that a subset of a compact metrizable space is an H-set if and only if it is simultaneously F_σ and G_δ, i.e. it is an *ambivalent set*, and that $\operatorname{ext} X$ is always G_δ if X is metrizable, see [1, Corollary I.4.4].) In [10], the conjecture was proved within a special class of simplices (so-called Stacey simplices).

In the present paper we prove that the conjecture is valid in full generality.

Let us now recall the definitions of some basic notions and fix some notation.

By a space we mean a topological Hausdorff space. If X is a compact space, we write $\mathcal{C}(X)$ for the space of all real-valued continuous functions on X. By $\mathcal{M}(X)$ we denote the set of all finite signed Radon measures on X endowed with the weak* topology. Recall that $\mathcal{M}(X)$ can be, due to Riesz's theorem, identified with the dual space $\mathcal{C}(X)^*$ and that the weak* topology is given by this identification. By $\mathcal{M}^+(X)$ we denote the positive measures from $\mathcal{M}(X)$, by $\mathcal{M}^1(X)$ we denote the probability measures from $\mathcal{M}(X)$. If $\mu \in \mathcal{M}(X)$ and $f : X \to \mathbb{R}$ is a μ-measurable function, we set $\mu(f) = \int_X f \, d\mu$.

Now suppose that X is a compact convex subset of a locally convex space. If f is a bounded function on X, its *upper envelope* f^* is defined as

$$f^*(x) = \inf\{h(x) : h \text{ continuous affine}, \ h \geqslant f\}, \quad x \in X.$$

We can see that f^* is the least upper semicontinuous concave function greater or equal than f. The *lower envelope* f_* is defined as $f_* := -(-f)^*$.

A point $x \in X$ is a *barycentre* of $\mu \in \mathcal{M}^1(X)$ if $f(x) = \mu(f)$ for each affine continuous function f on X. Any Radon probability measure on X has a unique barycentre which we denote by $r(\mu)$ (see [1, Proposition I.2.1]). A function $f : X \to \mathbb{R}$ is said to satisfy the *barycentric formula* if it is universally measurable and $\mu(f) = f(r(\mu))$ for each $\mu \in \mathcal{M}^1(X)$.

If $x \in X$, we say that a measure $\mu \in \mathcal{M}^1(X)$ is a *representing measure* for x if $x = r(\mu)$. The set of all probability measures representing x is denoted by $\mathcal{M}_x$. The classical Choquet–Bishop–de Leeuw theorem (see e.g. [1, Theorem I.4.8]) says that for any $x \in X$ there is a measure representing x which is maximal in the Choquet ordering. (Recall that $\mu \preccurlyeq \nu$ in the Choquet ordering if $\mu(f) \leqslant \nu(f)$ for each convex continuous function f on X.) If this maximal representing measure is unique for each $x \in X$, the set X is called a *Choquet simplex* (or, shortly, a *simplex*). In this case we denote by δ_x the unique maximal measure representing x. The Dirac measure supported by a point $x \in X$ is denoted by ε_x. We remark that $\delta_x = \varepsilon_x$ if and only if $x \in \operatorname{ext} X$.

If X is a simplex and f is a bounded universally measurable function on X, we can define

$$H^f(x) = \int_X f \, d\delta_x, \quad x \in X.$$

If f is a convex continuous function on X, $H^f = f^*$ (see Lemma 25). Since the differences of convex continuous functions on X are dense in $\mathcal{C}(X)$ by the lattice version of the Stone–Weierstrass theorem, H^f is a Borel function for each continuous, and consequently for each bounded Baire function f on X.

If f satisfies the barycentric formula, it is clearly affine. Conversely, affine continuous functions satisfy the barycentric formula by the definition of barycentre. Further, any affine Baire-one function on X is bounded and satisfies the barycentric formula (see e.g. [1, Theorem I.2.6]). This is not the case for general affine functions (even for Baire–two functions, see [1, Example I.2.10]).

2. Main theorem

The main result is contained in the following theorem.

Theorem 1. *Let X be a compact convex set. Then the following assertion are equivalent*:

(i) *X is a simplex and* $\operatorname{ext} X$ *is a Lindelöf H-set*;
(ii) *X is a simplex and for any closed G_δ-set $F \subset X$ the function $x \mapsto \delta_x(F)$, $x \in X$, is Baire-one*;
(iii) *X is a simplex and the function $x \mapsto \delta_x(f)$, $x \in X$, is Baire-one for every bounded Baire-one function f on X*;
(iv) *for every bounded Baire-one function f on X there exists an affine Baire-one function h on X, such that $f = h$ on* $\operatorname{ext} X$;

(v) *for every bounded Baire-one function f on* $\operatorname{ext} X$ *there exists an affine Baire-one function h on X such that $f = h$ on* $\operatorname{ext} X$.

Some of the implications are already known, the aim of the present paper is to prove the remaining ones. The implication (v) $\Longrightarrow$ (iv) is trivial. The equivalence of (iii) and (iv) is proved in [17, Corollary 1]. The equivalence of (ii) and (iii) follows from [10, Proposition 3].

In the present paper we will prove (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (i). The former is proved in Section 6, the latter in Sections 7 and 8.

This will close the chain of equivalences as the remaining implication (iv) $\Longrightarrow$ (v) follows from the implication (iv) $\Longrightarrow$ (i) and [11, Theorem 30].

The implication (i) $\Longrightarrow$ (ii) was proved by the first author in the preprint [18]. In the same preprint he proved that the assertion (iii) implies that $\operatorname{ext} X$ is an H-set. The remaining part, i.e., the fact that (iii) implies that $\operatorname{ext} X$ is Lindelöf, is a joint result of both authors contained in the preprint [12]. The present paper was done by merging these two preprints.

Note that the assertion (i) contains a topological property of $\operatorname{ext} X$, as promised in the abstract. Indeed, Lindelöf property is obviously a topological one. Moreover, a subset A of a compact space is an H-set if and only if each (nonempty) closed subset of A contains a dense locally compact subset. Or, equivalently, if and only if A admits a scattered partition to locally compact subsets.

3. Preliminaries

A function $f : X \to [-\infty, \infty)$ on a topological space X is said to be *upper semicontinuous* if the set $\{x \in X : f(x) < a\}$ is open in X for every $a \in \mathbb{R}$. A function f is *lower semicontinuous* if $-f$ is upper semicontinuous.

We will frequently use the following well-known results.

Theorem 2. *Let f be an upper semicontinuous function on a compact space X. Then the function*

$$\mu \mapsto \mu(f), \quad \mu \in \mathcal{M}^1(X),$$

is upper semicontinuous on $\mathcal{M}^1(X)$.

Proof. See [4, Theorem 30.10]. □

Lemma 3. *For any measure $\mu \in \mathcal{M}^+(X)$ on a compact space X and a closed subset K of X it holds*

$$\mu(K) = \inf\{\mu(\overline{W}) : W \supset K, W \text{ open}\}.$$

Proof. The assertion easily follows from the outer regularity of Radon measures and from the fact that, given an open set U containing K, there exists an open set W, such that $K \subset W \subset \overline{W} \subset U$. $\square$

We will also use the following version of the Monotone Convergence Theorem.

Theorem 4. *Let μ be a measure on a compact space X and $\mathcal{F}$ a downward directed family of upper semicontinuous functions on X (i.e. for any $f_1, f_2 \in \mathcal{F}$ there exists $f_3 \in \mathcal{F}$, such that $f_3 \leqslant \min\{f_1, f_2\}$). Then*

$$\mu\left(\inf_{f\in\mathcal{F}} f\right) = \inf_{f\in\mathcal{F}} \mu(f).$$

Proof. See [7, Theorem 12.46]. $\square$

In the sequel we will also need the following properties of Baire-one functions.

Proposition 5. *For a bounded real function f on a normal space X the following assertions are equivalent*:

(i) *f is a Baire-one function*;
(ii) *both sets $[f < a]$ and $[f > a]$ are of type F_σ for each $a \in \mathbb{R}$*;
(iii) *there exist sequences $\{f_n\}$, $\{g_n\}$ of functions on X, such that each f_n is upper semicontinuous, each g_n is lower semicontinuous,*

$$f_n \nearrow f \quad \text{and} \quad g_n \searrow f;$$

(iv) *for every $\varepsilon > 0$ there exists a partition $\{A_1, \dots, A_n\}$ of X consisting of ambivalent sets and real numbers $c_1, \dots, c_n$, such that*

$$\left\| f - \sum_{i=1}^{n} c_i \chi_{A_i} \right\|_\infty < \varepsilon.$$

Moreover, if X is a compact space, then the assertions above imply the following condition:

(v) *$f\restriction_H$ has a point of continuity for every closed nonempty $H \subset X$.*

Further, if X is a compact metric space, the condition (v) *is equivalent to the previous ones.*

Proof. See [14, Theorem 2.12 and Exercise 3.A.1] and [17, Theorem 2.1]. $\square$

The following definition is a quantified version of the (DP) condition from [14, Theorem 2.12].

Definition 6. Let f be a real function on a topological space X and $\varepsilon > 0$. We say that f satisfies the ε-(DP) *condition on* X if, for each nonempty closed set $H \subset X$ and every couple of real numbers $a < b$ with $b - a \geqslant \varepsilon$, the sets $[f \leqslant a] \cap H$ and $[f \geqslant b] \cap H$ are not simultaneously dense in H.

The following lemma closely follows the proof of Lukeš et al. [14, Theorem 2.12].

Lemma 7. *Let $\varepsilon > 0$ and let a bounded function f on a metrizable compact space X satisfy the ε-(DP) condition. Then for every real numbers $a < b$ with $b - a \geqslant \varepsilon$ there exists an ambivalent set H such that*

$$[f \leqslant a] \subset H \quad \textit{and} \quad [f \geqslant b] \subset X \setminus H.$$

Proof. Let $a < b$ be a couple of real numbers with $b - a \geqslant \varepsilon$. Let $\mathcal{G}$ be the family of all open sets $U \subset X$ for which there exists an ambivalent set $H_U \subset U$ satisfying

$$[f \leqslant a] \cap U \subset H_U \quad \text{and} \quad [f \geqslant b] \cap U \subset X \setminus H_U.$$

Obviously, if $U_1 \subset U_2$ and $U_2 \in \mathcal{G}$, then $U_1 \in \mathcal{G}$ as well. Set

$$G := \bigcup \mathcal{G}.$$

Since X is a metric space, we can find an open locally finite refinement $\mathcal{U}$ of $\mathcal{G}$ (see [13, §21.XVI]). Then

$$[f \leqslant a] \cap G \subset \bigcup_{U \in \mathcal{U}} H_U \quad \text{and} \quad [f \geqslant b] \cap G \subset X \setminus \bigcup_{U \in \mathcal{U}} H_U.$$

Since every locally finite union of ambivalent sets is again an ambivalent set (see [13, §30.X]), the set G belongs to $\mathcal{G}$.

If $G = X$, we are done. Otherwise we look at the set $F := X \setminus G$. Since the sets $[f \leqslant a]$ and $[f \geqslant b]$ cannot be simultaneously dense in F, there is an open set U intersecting F such that either

$$U \cap F \cap [f \leqslant a] = \emptyset \quad \text{or} \quad U \cap F \cap [f \geqslant b] = \emptyset.$$

In both cases $U \in \mathcal{G}$ which contradicts maximality of G (in the first case set $H_U := H_G \cap U$, and in the second case $H_U := (H_G \cap U) \cup (U \cap F)$).

Thus $G = X$ and the proof is finished. $\square$

Lemma 8. *Let $\varepsilon > 0$ and let a bounded function f on a metrizable compact space X satisfy the ε-(DP) condition. Then there exists a sequence $\{f_n\}$ of upper semicontinuous* *functions on X, such that*

$$\sup_n f_n \leqslant f \leqslant \sup_n f_n + 2\varepsilon.$$

Proof. Suppose that f has values in an interval $[a, b]$. By enlarging the interval $[a, b]$ if necessary we may assume that $b - a = n\varepsilon$ for some $n \in \mathbb{N}$. We set $a_k := a + k\varepsilon$, $k = 0, \ldots, n$. Using Lemma 7 we find ambivalent sets H_k, $k = 1, \ldots, n$, such that

$$[f \leqslant a_{k-1}] \subset H_k \quad \text{and} \quad [f \geqslant a_k] \subset X \setminus H_k.$$

For each $k \in \{1, \ldots, n\}$ we define

$$g_k(x) := \begin{cases} a_0, & x \in H_k, \\ a_k, & x \in X \setminus H_k. \end{cases}$$

Then $g_k = a_0$ on $[f \leqslant a_{k-1}]$ and $g_k = a_k$ on $[f \geqslant a_k]$. By setting

$$g := \max\{g_1, \ldots, g_n\},$$

we obtain a Baire-one function on X, such that $\|f - g\|_\infty \leqslant \varepsilon$.

Indeed, let $x \in X$ be given. Then $f(x) \in [a_{k-1}, a_k]$ for some $k \in \{1, \ldots, n\}$. Since $f(x) \geqslant a_{k-1}$, $g_{k-1}(x) = a_{k-1}$. Thus

$$f(x) \leqslant a_k = g_{k-1}(x) + \varepsilon \leqslant g(x) + \varepsilon.$$

On the other hand, $g_j(x) = a_0$ for each $j \in \{k+1, \ldots, n\}$ because $f(x) \leqslant a_k$. Since $f(x) \geqslant a_{k-1} \geqslant g_j(x) - \varepsilon$ if $j \in \{1, \ldots, k\}$,

$$f(x) \geqslant \sup\{g_j(x) : j = 1, \ldots, n\} - \varepsilon = g(x) - \varepsilon.$$

Thus $|f(x) - g(x)| \leqslant \varepsilon$ as needed.

Using Proposition 5 we find a sequence $\{f_n\}$ of upper semicontinuous functions on X, such that $g - \varepsilon = \sup_n f_n$. Then

$$\sup_n f_n = g - \varepsilon \leqslant f \leqslant g + \varepsilon = \sup_n f_n + 2\varepsilon.$$

Hence $\{f_n\}$ is the required sequence and we are done. $\square$

In the next lemma we need more details on the hierarchy of Baire functions on a topological space X. Let $\mathcal{F}$ be a family of real functions on a topological space X. We denote by $\mathcal{B}_1(\mathcal{F})$ the family of all pointwise limits of sequences consisting of functions from $\mathcal{F}$. Inductively, we define for each countable ordinal $\alpha \in (1, \omega_1)$ the family $\mathcal{B}_\alpha(\mathcal{F})$ to be the set of all pointwise limits of sequences of functions contained in the previous families.

If we take $\mathcal{F}$ to be the family $\mathcal{C}(X)$ of all continuous functions on X, we get the usual classes of Baire functions. It is easy to verify by transfinite induction that for any Baire function f of class α on X there exists a countable family $\mathcal{F} \subset \mathcal{C}(X)$ such that $f \in \mathcal{B}_\alpha(\mathcal{F})$.

Lemma 9. *Let $\varepsilon > 0$ and let a bounded Baire function f on a compact space X satisfy the ε-(DP) condition. Then there exists a sequence $\{g_n\}$ of upper semicontinuous functions on X, such that*

$$\sup_n g_n \leqslant f \leqslant \sup_n g_n + 2\varepsilon.$$

Proof. Given a bounded Baire function f on X, say of class α, let $\mathcal{F}$ be a countable family of continuous functions on X, such that $f \in \mathcal{B}_\alpha(\mathcal{F})$. We enumerate this family as $\{f_n : n \in \mathbb{N}\}$ and define a mapping

$$\varphi : X \to \mathbb{R}^{\mathbb{N}},$$
$$x \mapsto \{f_n(x)\}_{n\in\mathbb{N}}, \quad x \in X.$$

Then φ is a continuous mapping of X onto a metrizable compact space $Y := \varphi(X)$. Further, if $\varphi(x_1) = \varphi(x_2)$ for a couple of points $x_1, x_2 \in X$, then $f(x_1) = f(x_2)$. Thus we can define a function

$$\widehat{f} : Y \to \mathbb{R},$$
$$y \mapsto f(x), \quad x \in \varphi^{-1}(y), \quad y \in Y.$$

Then

$$f = \widehat{f} \circ \varphi$$

and $\widehat{f}$ satisfies the ε-(DP) condition on Y.

Indeed, assume that there exist a nonempty closed set $H \subset Y$ and reals $a < b$ with $b - a \geqslant \varepsilon$, such that

$$\overline{[\widehat{f} \leqslant a] \cap H} = \overline{[\widehat{f} \geqslant b] \cap H} = H.$$

Using Zorn's lemma we find a minimal (with respect to inclusion) closed set $F \subset X$, such that $\varphi(F) = H$. We need the following claim:

If D is a dense subset of H, then $\varphi^{-1}(D)$ is dense in F.

To verify it, pick a nonempty open set U in F. Since F is minimal, $\varphi(F \setminus U) \neq H$. We find a point

$$d \in D \cap (H \setminus \varphi(F \setminus U)).$$

Then $d \in \varphi(U)$, in other words, $\varphi^{-1}(d) \cap U \neq \emptyset$, and the claim is proved.

It follows from the claim that the preimages of $[\widehat{f} \leqslant a] \cap H$ and $[\widehat{f} \geqslant b] \cap H$ are dense in F. Since $f = \widehat{f} \circ \varphi$, both sets $[f \leqslant a]$ and $[f \geqslant b]$ are dense in F. But this contradicts our assumption on f.

Since we know that $\widehat{f}$ satisfies the ε-(DP) condition, we may apply Lemma 8 and get a sequence $\{\widehat{g}_n\}$ of upper semicontinuous functions on Y, such that

$$\sup_n \widehat{g}_n \leqslant \widehat{f} \leqslant \sup_n \widehat{g}_n + 2\varepsilon.$$

We finish the proof by setting $g_n := \widehat{g}_n \circ \varphi$, $n \in \mathbb{N}$. $\square$

Lemma 10. *Let $\varepsilon > 0$ and let f, f_n, g_n, $n \in \mathbb{N}$, be functions on a compact space X such that every f_n is upper semicontinuous, every g_n is lower semicontinuous,*

$$\sup_n f_n \leqslant f \leqslant \inf_n g_n \quad \textit{and} \quad \inf_n g_n - \sup_n f_n < \varepsilon.$$

Then f satisfies the ε-(DP) condition.

Proof. Assume that there exist a nonempty closed set $H \subset X$ and real numbers $a < b$ with $b - a \geqslant \varepsilon$, such that

$$\overline{[f \leqslant a] \cap H} = \overline{[f \geqslant b] \cap H} = H.$$

Then the sets

$$G_1 := \bigcap_{n=1}^{\infty} [f_n \leqslant a] \quad \text{and} \quad G_2 := \bigcap_{n=1}^{\infty} [g_n \geqslant b]$$

are of type G_δ,

$$[f \leqslant a] \subset G_1 \quad \text{and} \quad [f \geqslant b] \subset G_2.$$

Moreover, they are disjoint. Indeed, assuming that there is a point $x \in G_1 \cap G_2$, we obtain

$$b \leqslant \inf_n g_n(x) < \sup_n f_n(x) + \varepsilon \leqslant a + \varepsilon \leqslant b,$$

an obvious contradiction.

Thus G_1 and G_2 is a couple of nonempty disjoint G_δ-sets that are both dense in H. Since H is a Baire space, we have arrived to a contradiction and finished the proof. □

The following lemma is a particular case of one implication in [14, Theorem 2.12].

Lemma 11. *Let a Baire function f on a compact space X satisfy the ε-(DP) condition for every $\varepsilon > 0$. Then f is a Baire-one function.*

Proof. As f satisfies the ε-(DP) condition for every $\varepsilon > 0$, Lemma 9 provides a countable family $\mathcal{U}$ of upper semicontinuous functions on X, such that

$$\sup_{u \in \mathcal{U}} u = f.$$

Another application of Lemma 9 to the function $-f$ yields the existence of a countable family $\mathcal{L}$ of lower semicontinuous functions on X, such that

$$\inf_{l \in \mathcal{L}} l = f.$$

It is easy to construct a sequence $\{f_n\}$ of upper semicontinuous functions and a sequence $\{g_n\}$ of lower semicontinuous functions, such that

$$f_n \nearrow f \quad \text{and} \quad g_n \searrow f.$$

According to Proposition 5, the function f is Baire-one. □

A subset U of a topological space X is called *cozero* if $U = f^{-1}(0, 1]$ for some continuous function $f : X \to [0, 1]$. The smallest σ-algebra containing all cozero sets is the family of all *Baire* sets. We recall that any Baire subset of a compact space is Lindelöf (see [16, Section 2.7]).

Lemma 12. *Let C be a compact space, D a compact metric space and $\varphi : C \to D$ be a continuous surjection. Let g be a bounded Baire-one function on C. Then there exists a function $\psi : D \to C$, such that $\varphi(\psi(y)) = y$ for all $y \in D$ and $g \circ \psi$ is Baire-one on D.*

Proof. Let $\{g_n\}$ be a bounded sequence of Baire-one functions uniformly tending to g, such that each g_n is a simple function (see Proposition 5(iv)), i.e.

$$g_n = \sum_{i=1}^{k_n} c_{i,n} \chi_{A_{i,n}}, \quad n = 1, 2, \ldots,$$

where $c_{i,n} \in \mathbb{R}$ and every set $A_{i,n}$ is ambivalent. Write each set $A_{i,n}$ and $C \setminus A_{i,n}$ as a countable union of closed sets. By putting these closed sets together we obtain a countable family $\mathcal{F}$ of closed sets in C.

We apply [8, Lemma 8 and the subsequent Remark] to the family $\mathcal{F}$ in order to get a function ψ, such that $\varphi(\psi(y)) = y$ if $y \in D$ and $\psi^{-1}(F)$ is an ambivalent set in D for every $F \in \mathcal{F}$. Thus $g_n \circ \psi$ is Baire-one on D for every $n \in \mathbb{N}$. Since $\{g_n \circ \psi\}$ tends uniformly to $g \circ \psi$, the proof is finished. $\square$

An important ingredient of the proof of the main result is a characterization of Lindelöf subsets of compact spaces by a separation property (Lemma 15 below). So let us define what we mean by separation.

Definition 13. Let A and B be subsets of a space X and let $\mathcal{F}$ be a family of subsets of X. We say that A *can be separated from B by an $\mathcal{F}$-set* if there exists $F \in \mathcal{F}$, such that $A \subset F \subset X \setminus B$.

Lemma 14. *Let K and A be subsets of a compact space X, such that K is closed. Then the following assertions are equivalent*:

(i) *A can be separated from K by an F_σ-set.*
(ii) *A can be separated from K by a Baire set.*
(iii) *A can be separated from K by a Lindelöf set.*

Proof. For the proof of (i) $\Longrightarrow$ (ii), let F_n, $n \in \mathbb{N}$, be closed sets in X, such that $F := \bigcup_n F_n$ satisfies $A \subset F \subset X \setminus K$.

Fix $n \in \mathbb{N}$ and find for each $x \in K$ a cozero set U_x, such that $x \in U_x \subset X \setminus F_n$. By compactness there exist finitely many points $x_1, \ldots, x_k$ in K, such that $K \subset U_{x_1} \cup \cdots \cup U_{x_k}$. By setting $V_n := U_{x_1} \cup \cdots \cup U_{x_k}$ we get a cozero set with $K \subset V_n \subset X \setminus F_n$.

Then $X \setminus \bigcap_n V_n$ is a Baire set separating A from K.

The implication (ii) $\Longrightarrow$ (iii) follows from the fact that any Baire subset of a compact space is Lindelöf (see [16, Section 2.7]).

For the final implication (iii) $\Longrightarrow$ (i), let B be a Lindelöf set separating A from K. For any $x \in B$ find a cozero set U_x containing x and disjoint from K. Using the Lindelöf property we select countably many points $x_n \in B$, $n \in \mathbb{N}$, such that $B \subset \bigcup_n U_{x_n}$. Then the last set is an F_σ-set separating A from K. $\square$

Lemma 15. *Let X be a compact space and $A \subset X$. Then A is Lindelöf if and only if any compact subset of $X \setminus A$ can be separated from A by a G_δ-set.*

Proof. Then 'only if' part follows from Lemma 14, (iii) $\Longrightarrow$ (i). We will prove the 'if' part. We proceed similarly as in the proof of Kalenda [10, Lemma 4].

Let $\{U_a : a \in I\}$ be a covering of A consisting of relatively open sets. We are going to show that there is a countable subcover.

If there is a finite subcover, we are done. Otherwise we set $F_a = A \setminus U_a$ for $a \in I$ and we can see that the family $\{F_a : a \in I\}$ has finite intersection property and hence

$F = \bigcap_{a \in I} \overline{F}_a \neq \emptyset$ where the closures are taken in X. But each F_a is relatively closed in A and $\bigcap_{a\in I} F_a = \emptyset$ (as U_a's cover A). Therefore, $F \subset X \setminus A$ and hence F can be separated from A by a G_δ-subset of X.

Fix a sequence G_n of open subsets of X with $F \subset \bigcap_{n\in\mathbb{N}} G_n \subset X \setminus A$. For each $n \in \mathbb{N}$ there is a finite set $J_n \subset I$ with $\bigcap_{a\in J_n} F_a \subset G_n$. If we set $J = \bigcup_{n\in\mathbb{N}} J_n$, then J is a countable subset of I satisfying $\bigcap_{a\in J} F_a = \emptyset$ and hence $\bigcup_{a\in J} U_a = A$. $\square$

4. Auxiliary results on compact convex sets

Throughout this section we will assume that X is a compact convex set. We start with the following auxiliary notion.

Definition 16. Let F_0, F_1 be nonempty sets in X and $\eta > 0$. We say that the pair (F_0, F_1) is *η-singular* if for every $x_i \in F_i$ and $\mu_i \in \mathcal{M}_{x_i}$, $i = 0, 1$, it holds $\mu_0(F_1) < \eta$ and $\mu_1(F_0) < \eta$.

Remark 17. We point out that an η-singular pair (F_0, F_1) with $\eta \in (0, 1)$ consists of disjoint sets.

We also remark that (H_0, H_1) is η-singular if H_i's are nonempty, $H_i \subset F_i$, $i = 0, 1$, and (F_0, F_1) is η-singular.

Lemma 18. *Let f be an upper semicontinuous function on X. Then*

$$f^*(x) = \max\{\mu(f) : \mu \in \mathcal{M}_x\}, \quad x \in X.$$

Proof. See [1, Corollary I.3.6 and the subsequent Remark]. $\square$

Lemma 19. *The set*

$$R := \{(x, \mu) \in X \times \mathcal{M}^1(X) : \mu \textit{ represents } x\}$$

is closed in $X \times \mathcal{M}^1(X)$.

Proof. Let $\{(x_\alpha, \mu_\alpha)\}$ be a net in R converging to (x, μ). If h is an affine continuous function on X, $\mu_\alpha(h) = h(x_\alpha)$ for all α due to the definition of R. By passing to the limit in the equality above yields $\mu(h) = h(x)$. Thus $\mu \in \mathcal{M}_x$ as needed. $\square$

Lemma 20. *Let $\eta > 0$ and let (F_0, F_1) be an η-singular pair of closed sets. Then there exist open sets U_0, U_1, such that $F_i \subset U_i$, $i = 0, 1$, and $(\overline{U}_0, \overline{U}_1)$ is η-singular.*

Proof. Let R be the set from Lemma 19. By this lemma and Theorem 2 the set

$$K := \{(x_0, \mu_0, x_1, \mu_1) \in R \times R : \mu_0(F_1) \geqslant \eta \text{ or } \mu_1(F_0) \geqslant \eta\}$$

is compact. Hence its projection L onto $X \times X$ is compact and disjoint from $F_0 \times F_1$ by η-singularity. Another use of compactness yields the existence of open sets U_0, U_1 in X, such that

$$F_0 \times F_1 \subset \overline{U}_0 \times \overline{U}_1 \subset (X \times X) \setminus L.$$

(Indeed, first we find for any $x \in F_1$ open sets U_x, V_x, such that

$$F_0 \times \{x\} \subset \overline{U}_x \times \overline{V}_x \subset (X \times X) \setminus L.$$

By compactness we can find finitely many open sets $U_1, \dots, U_n$ and $V_1, \dots, V_n$, such that each U_i contains F_0 and

$$F_0 \times F_1 \subset \bigcup_{i=1}^{n} (\overline{U}_i \times \overline{V}_i) \subset (X \times X) \setminus L.$$

Then $U_1 \cap \dots \cap U_n$ and $V_1 \cup \dots \cup V_n$ are the required open sets.)

Then $(\overline{U}_0, \overline{U}_1)$ is the sought pair of η-singular sets. □

Lemma 21. *Let μ, ν be measures on X with $\mu \prec \nu$. Then $\mu(f) \leqslant \nu(f)$ for every upper semicontinuous convex function f on X. In particular, $\mu(K) \leqslant \nu(K)$ for every compact $K \subset \operatorname{ext} X$.*

Proof. The first part easily follows from the definition of the Choquet ordering, [2, Theorem 6.1(x)] and Theorem 4. As the function χ_K is upper semicontinuous and convex provided $K \subset \operatorname{ext} X$ is compact, the lemma is proved. □

Lemma 22. *Let h satisfy the barycentric formula on X and f, g be functions such that f is upper semicontinuous, g is lower semicontinuous and*

$$f \leqslant h \leqslant g \quad (\textit{respectively}, \ \ f < h < g).$$

Then

$$f^* \leqslant h \leqslant g_* \quad (\textit{respectively}, \ \ f^* < h < g_*).$$

Proof. Let f be an upper semicontinuous function on X, such that $f \leqslant h$ and $x \in X$ be arbitrary. According to Lemma 18, there exists a measure $\mu \in \mathcal{M}_x$ such that

$f^*(x) = \mu(f)$. Thus

$$f^*(x) = \mu(f) \leqslant \mu(h) = h(x),$$

as h satisfies the barycentric formula.

Since the remaining cases can be treated similarly, the proof is finished. □

Lemma 23. *For any point $x \in \operatorname{ext} X$, an open set $U \ni x$ and $\eta > 0$ there exists a neighbourhood V of x, such that $\mu(U) > 1 - \eta$ for every μ representing a point $y \in V$.*

Proof. Suppose that there exists a point $x \in \operatorname{ext} X$, an open set U containing x and $\eta > 0$, such that for each open $V \ni x$ there is a point $x_V \in V$ and a measure $\mu_V \in \mathcal{M}_{x_V}$, such that $\mu_V(U) < 1 - \eta$. Without loss of generality we may assume that $\{\mu_V\}$ converges to a probability measure μ on X. Theorem 2 yields

$$\mu(U) \leqslant \liminf_V \ \mu_V(U) \leqslant 1 - \eta.$$

On the other hand, $\lim_V x_V = x$, which gives that the measure μ represents x (see Lemma 19). Since the only representing measure for an extreme point is the Dirac measure, we get a contradiction with the estimation above. This concludes the proof. □

Lemma 24. *Let $\operatorname{ext} X$ be Lindelöf, H a Borel subset of X, $\varepsilon > 0$, f a bounded Borel function on H and $\{f_n\}$ a sequence of Borel functions, such that*

$$\sup_n \ f_n \leqslant f \leqslant \sup_n \ f_n + \varepsilon \quad \text{on } H \cap \operatorname{ext} X.$$

Then for every maximal measure μ holds

$$\int_H (f - \varepsilon)\, d\mu \leqslant \int_H \left(\sup_n f_n \right) d\mu \leqslant \int_H f \, d\mu.$$

Proof. Given a maximal measure μ, we define

$$B := \left\{ x \in H : \sup_n \ f_n(x) + \varepsilon < f(x) \right\}.$$

Then B is a Borel set disjoint from $\operatorname{ext} X$. We claim that $\mu(B) = 0$.

Indeed, let $K \subset X$ be a compact set disjoint from $\operatorname{ext} X$. As $\operatorname{ext} X$ is Lindelöf, by Lemma 15 there is a G_δ-set $G \subset X$, such that $K \subset G \subset X \setminus \operatorname{ext} X$. Since μ is supported by any F_σ-set containing $\operatorname{ext} X$ (see [1, Corollary I.4.12 and the subsequent

Remark]), $\mu(G)=0$ and hence $\mu(K)=0$. Finally, the regularity of μ yields

$$\mu(B)=\sup\{\mu(K): K\subset B,\, K \text{ compact}\}=0.$$

Hence

$$\begin{aligned}\int_H \left(\sup_n f_n\right) d\mu &= \int_{H\setminus B} \left(\sup_n f_n\right) d\mu\\ &\geqslant \int_{H\setminus B} (f-\varepsilon)\, d\mu\\ &= \int_H (f-\varepsilon)\, d\mu.\end{aligned}$$

Since the second required inequality can be proved in a similar manner, the proof is finished. □

5. Auxiliary results on simplices

Throughout this section we will assume that X is a simplex. We recall that for any bounded Baire function f on X the function H^f is defined as

$$H^f(x)=\delta_x(f),\quad x\in X.$$

Lemma 25. *Let f be an upper semicontinuous convex function on X. Then $f^*=H^f$ and H^f satisfies the barycentric formula. In particular, $\chi_K^*=H^{\chi_K}$ for any compact set $K\subset \operatorname{ext} X$.*

Proof. For a given point $x\in X$ we find a measure $\mu\in\mathcal{M}_x$, such that $\mu(f)=f^*(x)$ (see Lemma 18). Since $\mu\prec\delta_x$, Lemmas 18 and 21 yield

$$\delta_x(f)\leqslant f^*(x)=\mu(f)\leqslant\delta_x(f).$$

Thus $f^*=H^f$.

It follows from [1, Proposition I.4.5] that positive maximal measures on X form a cone. Since X is a simplex, the mapping $x\mapsto\delta_x$ is affine. Thus H^f is affine as well. According to [1, Corollary I.1.4] and Theorem 4, any semicontinuous affine function satisfies the barycentric formula.

As χ_K is upper semicontinuous convex for every compact set $K\subset\operatorname{ext} X$, the last assertion is a consequence of the first part. This finishes the proof. □

Lemma 26. *Let* ext X *be Lindelöf and f be a bounded Baire function on X. Then* H^f *is a Baire function as well.*

Proof. If f is a continuous function on X, the function H^f is Baire-one due to [9, Theorem]. If $\mathcal{F}$ denotes the family of all bounded Baire functions f on X such that H^f is a Baire function, we get a family closed with respect to taking pointwise limits of converging bounded sequences. As $\mathcal{C}(X) \subset \mathcal{F}$, the family $\mathcal{F}$ contains any bounded Baire function on X. □

Lemma 27. *Let f be a bounded Baire function on X. Then* H^f *satisfies the barycentric formula.*

Proof (*cf. Kalenda [10, Proposition 8]*). Let $\mathcal{F}$ denote the family of all bounded Baire functions f, such that H^f satisfies the barycentric formula. Obviously, $\mathcal{F}$ is closed with respect to taking pointwise limits of converging bounded sequences and $\mathcal{F}$ contains $\mathcal{C}(X)$.

Indeed, if f is a convex continuous function on X, $H^f = f^*$ satisfies the barycentric formula due to Lemma 25. According to the Stone–Weierstrass theorem, $\mathcal{C}(X) \subset \mathcal{F}$. Thus $\mathcal{F}$ contains all bounded Baire functions and we are through. □

Lemma 28. *Let K be a compact subset of* ext X, $D \subset X$ *closed and* $\varepsilon > 0$. *Then there exists an open set U containing K, such that*

$$D_U := \{x \in D : \delta_x(U \setminus K) \geqslant \varepsilon\}$$

is not dense in D.

Proof. Let K, D and $\varepsilon > 0$ be as in the statement. Assume that for every open $U \supset K$ the set D_U is dense in D. We will prove the following claim:

For every $n \in \mathbb{N} \cup \{0\}$ *and every* $x \in D$ *it holds* $\delta_x(K) \geqslant n\varepsilon$.

Once we accomplish this task we will get an obvious contradiction.

The proof of the claim will be done by induction. As the inequality $\delta_x(K) \geqslant 0$ for each $x \in D$ is obvious, we proceed to the inductive step.

Assume that the claim holds true for some $n \in \mathbb{N} \cup \{0\}$. Fix an arbitrary point $x \in D$. It follows from our assumption that, for every open $U \supset K$ and any open $V \ni x$ we can find a point $x_{U,V} \in D$, such that $x_{U,V} \in V$ and $\delta_{x_{U,V}}(U \setminus K) \geqslant \varepsilon$. By passing to a subnet if necessary we may assume that $\{\delta_{x_{U,V}}\}$ converges to a probability measure μ on X. Since $x_{U,V} \to x$, $\mu \in \mathcal{M}_x$ by Lemma 19.

Let W be an open set containing K. Using the inductive assumption and Theorem 2 we get

$$\begin{aligned}\mu(\overline{W}) &\geqslant \limsup_{U,V} \delta_{x_{U,V}}(\overline{W})\\ &\geqslant \limsup_{U,V} \delta_{x_{U,V}}(U)\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{U,V} \big(\delta_{x_{U,V}}(U \setminus K) + \delta_{x_{U,V}}(K)\big) \\
&\geqslant \limsup_{U,V} \delta_{x_{U,V}}(U \setminus K) + n\varepsilon \\
&\geqslant (n+1)\varepsilon.
\end{aligned}
$$

According to Lemma 3, $\mu(K) \geqslant (n+1)\varepsilon$. As $\mu \in \mathcal{M}_x$, $\mu \preccurlyeq \delta_x$. Further, χ_K is an upper semicontinuous convex function on X and thus $\mu(K) \leqslant \delta_x(K)$ (see Lemma 21). Hence

$$
(n+1)\varepsilon \leqslant \mu(K) \leqslant \delta_x(K).
$$

As x is arbitrary, the proof of the claim is finished as well as the proof of the lemma. □

6. Proof of (i) $\Longrightarrow$ (ii)

We are going to prove the following proposition.

Proposition 29. *Let X be a simplex such that the set* $\operatorname{ext} X$ *is Lindelöf. If* $\operatorname{ext} X$ *is an H-set, then for any closed G_δ-set $F \subset X$ the function H^{χ_F} is Baire-one.*

Proof. Assume that there is a closed G_δ-set F, such that H^{χ_F} is not a Baire-one function. Obviously, F is nonempty. We will find a nonempty closed set H such that

$$
\overline{H \cap \operatorname{ext} X} = \overline{H \setminus \operatorname{ext} X} = H, \tag{1}
$$

which will show that $\operatorname{ext} X$ is not an H-set.

We set

$$
f(x) := H^{\chi_F}(x), \quad x \in X.
$$

Then f is a Baire function (see Lemma 26), satisfies the barycentric formula (see Lemma 27) and $f(X) \subset [0,1]$ (obvious). Note also that $f = 1$ on $\operatorname{ext} X \cap F$ and $f = 0$ on $\operatorname{ext} X \cap (X \setminus F)$.

We begin the first part of the proof with the following couple of lemmas.

Lemma 29.1. *There exists a decreasing sequence $\{a_n\}$ of affine lower semicontinuous functions on X, such that $f = \inf_n a_n$ on X.*

Proof. Since F is closed and G_δ, there exists a decreasing sequence $\{g_n\}$ of continuous functions on X such that $g_n \searrow \chi_F$.

Fix $n \in \mathbb{N}$. For every $x \in \operatorname{ext} X$, Lemma 18 yields $g_n(x) = g_n^*(x)$. If a continuous affine function h with $h \geqslant g_n$ satisfies $h(x) - g_n(x) < \varepsilon$, the same inequality holds also for some neighbourhood of x. Since $\operatorname{ext} X$ is Lindelöf, for every $\varepsilon > 0$ we are able to select a countable family $\mathcal{A}_{n,\varepsilon}$ of affine continuous functions such that $\inf\{h : h \in \mathcal{A}_{n,\varepsilon}\} \leqslant g_n + \varepsilon$ on $\operatorname{ext} X$. By setting $\varepsilon_k := \frac{1}{k}$, $k \in \mathbb{N}$, and putting all the respective families together, we obtain a countable family $\mathcal{A}_n$ of affine continuous functions satisfying $g_n = \inf\{h : h \in \mathcal{A}_n\}$ on $\operatorname{ext} X$.

Since $g_n \searrow f$ on $\operatorname{ext} X$, the family $\mathcal{A} = \bigcup_n \mathcal{A}_n$ satisfies

$$f = \inf_{h \in \mathcal{A}} h \quad \text{on } \operatorname{ext} X.$$

We enumerate the family $\mathcal{A}$ as a sequence $\{h_n\}$ and inductively define $f_1 = h_1$, $f_2 = \min\{f_1, h_2\}, \ldots$. Then $\{f_n\}$ is a decreasing sequence of continuous concave functions such that $f = \lim_n f_n$ on $\operatorname{ext} X$.

If we set

$$a_n := H^{f_n}, \quad n \in \mathbb{N},$$

we obtain a decreasing sequence of lower semicontinuous affine functions on X (see Lemma 25) converging to f on $\operatorname{ext} X$. Using Lemma 24 and the Monotone Convergence Theorem we conclude that $f = \lim_n a_n$ on X (Apply Lemma 24 for $H := X, -f$, $-a_n$ and arbitrary $\varepsilon > 0$). $\square$

Lemma 29.2. *Let H_1, H_2 be nonempty closed subsets of X, such that $H_1 \subset H_2$ and $\varepsilon_1, \varepsilon_2 \geqslant 0$. Let $\{f_n\}$ be a sequence of upper semicontinuous functions on X, such that*

$$\sup_n f_n \leqslant f \leqslant \sup_n f_n + \varepsilon_1 \quad \textit{on } \operatorname{ext} X \cap H_2.$$

Assume that $\delta_x(H_2) \geqslant 1 - \varepsilon_2$ for every $x \in H_1$. Then the function $f\restriction_{H_1}$ satisfies the ε-(DP) condition for every $\varepsilon > \varepsilon_1 + \varepsilon_2$.

Proof. Given a sequence $\{f_n\}$ satisfying the assumptions above, we may assume that each f_n is positive and $\{f_n\}$ is increasing. Pick an arbitrary $x \in H_1$. Since δ_x is maximal,

$$\int_{H_2} \left(\lim_n f_n\right) d\delta_x \geqslant \int_{H_2} (f - \varepsilon_1)\, d\delta_x \tag{2}$$

due to Lemma 24. Since f satisfies the barycentric formula (see Lemma 27), Lemma 22 gives that $f_n^*(x) \leqslant f(x)$ for every $n \in \mathbb{N}$. A consecutive application of this fact, Lemma 18, the Monotone Convergence Theorem, inequality (2) and our assumptions yields

$$\begin{aligned}
f(x) &\geqslant \lim_n f_n^*(x) \\
&\geqslant \lim_n \int_X f_n \, d\delta_x \\
&= \int_X \left(\lim_n f_n\right) d\delta_x \\
&= \int_{X\setminus H_2} \left(\lim_n f_n\right) d\delta_x + \int_{H_2} \left(\lim_n f_n\right) d\delta_x \\
&\geqslant 0 + \int_{H_2} (f - \varepsilon_1) \, d\delta_x \\
&= \int_X f \, d\delta_x - \varepsilon_1 \delta_x(H_2) - \int_{X\setminus H_2} f \, d\delta_x \\
&\geqslant f(x) - \varepsilon_1 - \varepsilon_2.
\end{aligned}$$

Hence we obtain that

$$\sup_n f_n^* \leqslant f \leqslant \sup_n f_n^* + (\varepsilon_1 + \varepsilon_2) \quad \text{on } H_1.$$

Let $\{g_n\}$ be a sequence provided by Lemma 29.1. Then

$$\inf_n g_n - \sup_n f_n^* \leqslant \varepsilon_1 + \varepsilon_2 \quad \text{on } H_1.$$

Using Lemma 10 we conclude the proof. □

For every $\varepsilon > 0$ we define

$$\begin{aligned}
G_\varepsilon &:= \{x \in F : \text{there exists an open } U \ni x, \text{ such that} \\
&\qquad f\restriction_{U\cap F} \text{ satisfies the } \varepsilon\text{-(DP) condition}\}, \\
F_\varepsilon &:= F \setminus G_\varepsilon.
\end{aligned}$$

Then every G_ε is an open set in F and every F_ε is closed.

Lemma 29.3. *There exists an $\varepsilon > 0$ such that $F_\varepsilon \cap \operatorname{ext} X \neq \emptyset$.*

Proof. Suppose that $F_\varepsilon \cap \operatorname{ext} X = \emptyset$ for every $\varepsilon > 0$. Then for each $x \in F \cap \operatorname{ext} X$ we can find a closed neighbourhood K_x, such that $f\restriction_{(K_x \cap F)}$ satisfies the ε-(DP) condition. As $F \cap \operatorname{ext} X$ is Lindelöf, we select countably many compact sets K_n, $n \in \mathbb{N}$, whose union covers $F \cap \operatorname{ext} X$ and the restriction of f to $K_n \cap F$ satisfies the ε-(DP) condition. According to Lemma 9, for every $n \in \mathbb{N}$ there exists a countable family $\{f_{n,k} : k \in \mathbb{N}\}$ of positive upper semicontinuous functions on K_n, such that

$$\sup_k f_{n,k} \leqslant f \leqslant \sup_k f_{n,k} + 2\varepsilon \quad \text{on } K_n \cap F.$$

Extend the functions $f_{n,k}$, $k \in \mathbb{N}$ to X by 0 on $X \setminus K_n$. Then $\mathcal{U} := \{f_{n,k} : n, k \in \mathbb{N}\}$ is a countable family of upper semicontinuous functions, such that

$$\sup_{u \in \mathcal{U}} u \leqslant f \leqslant \sup_{u \in \mathcal{U}} u + 2\varepsilon \quad \text{on } \operatorname{ext} X.$$

It follows from Lemma 29.2 that f satisfies the η-(DP) condition for every $\eta > 2\varepsilon$ (take $H_1 = H_2 = X$, $\varepsilon_1 = 2\varepsilon$ and $\varepsilon_2 = 0$).

As $\varepsilon > 0$ is arbitrary, f satisfies the η-(DP) condition on X for every $\eta > 0$. Lemma 11 yields that f is a Baire-one function on X which contradicts our assumption. □

We define the required closed set H as

$$H := \overline{\bigcup_{\varepsilon > 0} (F_\varepsilon \cap \operatorname{ext} X)}.$$

Then $H \subset F$, and by Lemma 29.3, H is nonempty. Obviously, $H \cap \operatorname{ext} X$ is a dense subset of H. It remains to verify that $H \setminus \operatorname{ext} X$ is dense in H as well. To this end, it is enough to prove that, given $\varepsilon_1 > 0$, $x_1 \in F_{\varepsilon_1} \cap \operatorname{ext} X$ and a neighbourhood V of x_1 (in H), there exists a point in $V \cap (H \setminus \operatorname{ext} X)$.

Assume that this is not the case. Thus there exists a closed neighbourhood V of x_1, such that

$$K := V \cap H \subset \operatorname{ext} X.$$

We pick $\varepsilon_2 \in (0, \varepsilon_1/4)$. Using Lemma 23 we find a closed neighbourhood W of x_1, such that $W \subset V$ and $\delta_x(V) \geqslant 1 - \varepsilon_2$ for every $x \in W$. As $x_1 \in F_{\varepsilon_1}$, there exists a nonempty closed set $D \subset W \cap F$ and a couple of real numbers $a < b$ with $b - a \geqslant \varepsilon_1$, such that

$$\overline{[f \leqslant a] \cap D} = \overline{[f \geqslant b] \cap D} = D. \tag{3}$$

Obviously, we may assume that D does not intersect K as $f = 1$ on K.

Choose a number $\varepsilon_3 \in (0, \varepsilon_1/4)$ and use Lemma 28 to find an open set $U \supset K$, such that the set

$$D_U := \{x \in D : \delta_x(U \setminus K) \geqslant \varepsilon_3\}$$

is not dense in D. Since $D \cap K = \emptyset$, we may also achieve that $D \cap U = \emptyset$ by a suitable adjustment of U. It follows that we can shrink the set D in such a way that D still satisfies (3) and

$$\delta_x(U \setminus K) \leqslant \varepsilon_3$$

for every $x \in D$.

Up to now we have obtained a nonempty closed set D in $(W \cap F) \setminus U$, such that

$$f\restriction_D \text{ violates the } \varepsilon_1\text{-(DP) condition} \tag{4}$$

and, for every $x \in D$, we have

$$\delta_x((V \setminus U) \cup K) \geqslant \delta_x(V) - \delta_x(U \setminus K) \geqslant 1 - \varepsilon_2 - \varepsilon_3. \tag{5}$$

Choose $\varepsilon_4 \in (0, \varepsilon_1/8)$. Since for every $\varepsilon > 0$

$$(V \setminus U) \cap F \cap \operatorname{ext} X \subset G_\varepsilon$$

for each

$$x \in (V \setminus U) \cap F \cap \operatorname{ext} X$$

we can find a closed neighbourhood G_x of x, such that $f\restriction_{G_x \cap F}$ satisfies the ε_4-(DP) condition. Using the Lindelöf property of $(V \setminus U) \cap F \cap \operatorname{ext} X$ we select countably many compact sets K_n, $n \in \mathbb{N}$, such that their union covers $(V \setminus U) \cap F \cap \operatorname{ext} X$ and $f\restriction_{K_n \cap F}$ satisfies the ε_4-(DP) condition. According to Lemma 9, there are countable families $\mathcal{U}_n$ of positive upper semicontinuous functions on $K_n \cap F$, such that

$$\sup_{u \in \mathcal{U}_n} u \leqslant f\restriction_{K_n \cap F} \leqslant \sup_{u \in \mathcal{U}_n} u + 2\varepsilon_4.$$

Extend every function $u \in \mathcal{U}_n$ to X by setting $u := 0$ on $X \setminus (K_n \cap F)$.

Set

$$\mathcal{U} := \bigcup_{n=1}^{\infty} \mathcal{U}_n \cup \{\chi_K\}.$$

Then $\mathcal{U}$ is a countable family of upper semicontinuous functions on X, such that

$$\sup_{u \in \mathcal{U}} u \leqslant f \leqslant \sup_{u \in \mathcal{U}} u + 2\varepsilon_4 \quad \text{on } ((V \setminus U) \cup K) \cap \operatorname{ext} X. \tag{6}$$

We apply Lemma 29.2 to closed sets $H_1 := D$, $H_2 := (V \setminus U) \cup K$ and the countable family $\mathcal{U}$. Then conditions (5) and (6) gives that the restriction of f to the set D satisfies the ε-(DP) condition for every $\varepsilon > \varepsilon_2 + \varepsilon_3 + 2\varepsilon_4$. Since our choice of ε_2, ε_3, ε_4 ensures that

$$\varepsilon_1 > \varepsilon_2 + \varepsilon_3 + 2\varepsilon_4,$$

we have arrived to a contradiction with (4). Thus, our assumption that $H \setminus \operatorname{ext} X$ is not dense in H is false and H is the sought nonempty closed set satisfying (1). This finishes the proof. $\square$

7. Condition (iii) implies the Lindelöf property of ext X

The aim of this section is to prove a series of lemmas and propositions which enables us to show that $\operatorname{ext} X$ is Lindelöf whenever (iii) of Theorem 1 holds. We will need the following notion of local separation.

Definition 30. For a point $x \in A$ we say that A *can be locally separated by an $\mathcal{F}$-set from B at x* if there exists an open $U \subset X$ containing x, such that $U \cap A$ can be $\mathcal{F}$-separated from B.

Lemma 31. *Let K and A be subsets of a compact space X, such that K is a closed set that cannot be separated from A by a G_δ-set. Then there exists a nonempty closed subset L of K, such that L cannot be locally separated from A by a Baire set at any point $x \in L$.*

Proof. Let K and A be as in the premise. According to Lemma 14, K is not separated from A even by a Baire set.

Set

$$G := \{x \in K : K \text{ can be locally separated from } A \text{ by a Baire set at } x\},$$

$$L := K \setminus G.$$

We can see that L is nonempty because otherwise we could use compactness to pick finitely many open sets U_i, $i = 1, \ldots, n$, and Baire sets B_i, $i = 1, \ldots, n$, such that $K \cap U_i \subset B_i \subset X \setminus A$ for each $i = 1, \ldots, n$ and U_i's cover K. In this case, $B_1 \cup \cdots \cup B_n$ would be a Baire set separating K from A, a contradiction with our assumption.

To finish the proof it is enough to show that L cannot be locally separated from A by a Baire set at any point. Assuming that this is not the case, there exists a point $x \in L$ together with a cozero set $U \subset X$ and a Baire set B such that $x \in U$ and $L \cap U \subset B \subset X \setminus A$. Then $K \cap (U \setminus B)$, as a Baire subset of a compact space K, is Lindelöf and it is contained in G. Thus, for each $y \in K \cap (U \setminus B)$ we can find an open set $U_y \ni y$ and a Baire set B_y such that $U_y \cap K \subset B_y \subset X \setminus A$. Using the Lindelöf property we find a Baire set C separating $K \cap (U \setminus B)$ from A. Then $C \cup B$ separates $K \cap U$ from A and thus $x \in G$. This contradiction finishes the proof. $\square$

Definition 32. Let (iii) of Theorem 1 hold for a compact convex set X. Let K be a nonempty compact subset of X disjoint from $\operatorname{ext} X$ such that K cannot be locally separated by a Baire set from $\operatorname{ext} X$ at any point $x \in K$. Throughout this section we will refer to this situation as (*) and call the compact set K *perfectly unseparable*. Our aim is to show that this situation leads to a contradiction.

Lemma 33. *In the situation of* (*), *if* $U \subset X$ *is an open set intersecting* K *and* $c \in (0, 1)$, *then the set*

$$D_{U,c} := \{x \in K \cap U : \delta_x(U) < 1 - c\}$$

is nowhere dense in K.

Proof. First we prove that $D_{U,c}$ has empty interior in K. Assuming the contrary, there exists a cozero set $V \subset U$ intersecting K, such that $V \cap K \subset D_{U,c}$. According to our assumption, the function $x \mapsto \delta_x(V)$ is a Baire-one function on X. Thus the set

$$B := V \cap \{x \in X : \delta_x(V) < 1 - c\}$$

is Baire and $V \cap K \subset B$. Moreover, $B \cap \operatorname{ext} X = \emptyset$. Indeed, for any $x \in \operatorname{ext} X \cap V$ we have $\delta_x = \varepsilon_x$ and thus $\delta_x(V) = 1$, i.e., x is not in B.

Hence B is a Baire set separating $V \cap K$ from $\operatorname{ext} X$. But this is impossible as K cannot be locally separated from $\operatorname{ext} X$ by a Baire set at any point.

To show that $D_{U,c}$ is nowhere dense in K, we again assume that this is not the case. Then we are able to find a cozero set $V \subset U$ intersecting K, such that $D_{U,c}$ is dense in $V \cap K$. We know from the previous paragraph that $D_{V,\frac{c}{2}}$ has empty interior in $K \cap V$. In other words, the set

$$\left\{x \in K \cap V : \delta_x(V) \geqslant 1 - \frac{c}{2}\right\}$$

is dense in $V \cap K$. Then the function $x \mapsto \delta_x(V)$, $x \in X$, is not Baire-one as both the sets

$$\{x \in K \cap V : \delta_x(V) < 1 - c\} \quad \text{and} \quad \left\{x \in K \cap V : \delta_x(V) \geqslant 1 - \frac{c}{2}\right\}$$

are dense in $\overline{K \cap V}$ (see Proposition 5(v)). But this contradicts our assumption (iii) of Theorem 1 and concludes the proof. $\square$

Lemma 34. *In the situation of* (*), *if* $U \subset X$ *is an open set intersecting* K *and* $c \in (0, 1)$, *then there exists an open set* $V \subset U$ *intersecting* K, *such that* $\delta_x(U) \geqslant 1 - c$ *for every* $x \in V \cap K$.

Proof. Given an open set U intersecting K and $c \in (0, 1)$, the set $D_{U,c} = \{x \in K \cap U : \delta_x(U) < 1 - c\}$ is nowhere dense in K by the previous lemma. Hence we can find an open set $V \subset U$, such that $V \cap K$ is nonempty and does not intersect $D_{U,c}$. Thus for every $x \in V \cap K$ we have $\delta_x(U) \geqslant 1 - c$. $\square$

Definition 35. Let X be a simplex. We define an operator $T : \mathcal{M}(X) \to \mathcal{M}(X)$ by the following formula:

$$T\mu(f) = \int_X \delta_x(f)\, d\mu(x), \quad f \in \mathcal{C}(X).$$

As it was mentioned in the introduction, the mapping $x \mapsto \delta_x(f)$, $x \in X$, is a bounded Borel function for each $f \in \mathcal{C}(X)$ and thus the integral is well defined. Moreover, it is easy to check that $T\mu$ is a bounded linear functional on $\mathcal{C}(X)$ and hence it can be represented as an element of $\mathcal{M}(X)$.

Lemma 36. *The operator* T *has the following properties*:

(i) T *is a positive linear operator*;
(ii) $T\mu(f) = \mu(f^*)$ *for any positive measure* μ *and any continuous convex function* f *on* X;
(iii) $\mu \preccurlyeq T\mu$ *and* $T\mu$ *is maximal for any positive measure* $\mu \in \mathcal{M}^+(X)$;
(iv) *for any positive measure* $\mu \neq 0$ *we have* $T\mu = \mu(X)\delta_x$, *where* x *is the barycentre of the probability* $\frac{\mu}{\mu(X)}$.

Proof. (i) The linearity is obvious. To see that T is positive recall that $\mu \in \mathcal{M}(X)$ is positive if and only if $\mu(f) \geqslant 0$ for any $f \in \mathcal{C}(X)$, $f \geqslant 0$. Obviously, $T\mu(f) \geqslant 0$ whenever $\mu \geqslant 0$ and $f \geqslant 0$.

(ii) It follows directly from Lemma 25.

(iii) Let f be a convex continuous function on X. As $f^* \geqslant f$, it follows from (ii) that

$$T\mu(f) = \mu(f^*) \geqslant \mu(f).$$

Hence $\mu \preccurlyeq T\mu$.

Further, by the definition we have

$$f^*(x) = \inf\{h(x) : h \text{ is continuous and affine on } X, h \geqslant f \text{ on } X\}, \quad x \in X.$$

By Edwards [6, Theorem 3] (see also [1, Theorem II.3.10]), the set of all affine continuous functions h on X, such that $h \geqslant f$ is downward directed. It follows that

$$\begin{aligned} T\mu(f^*) &= \inf\{T\mu(h) : h \text{ is continuous and affine on } X, h \geqslant f \text{ on } X\} \\ &= \inf\{\mu(h) : h \text{ is continuous and affine on } X, h \geqslant f \text{ on } X\} \\ &= \mu(f^*) \\ &= T\mu(f). \end{aligned}$$

The first and the third equality follows from Theorem 4. The second one is a consequence of the fact that $T\mu(h) = \mu(h)$ for each affine continuous function h on X (as $\mu \preccurlyeq T\mu$). The last equality follows from (ii).

Since $T\mu(f^*) = T\mu(f)$ for each continuous convex function, $T\mu$ is maximal by Alfsen [1, Proposition I.4.5].

(iv) Given a positive measure $\mu \neq 0$, $v = \frac{\mu}{\mu(X)}$ is a probability measure. By (i) we have $T\mu = \mu(X)Tv$. Let x denotes the barycentre of v. As $v \preccurlyeq Tv$, the barycentre of Tv is also x. Finally, as Tv is maximal, necessarily $Tv = \delta_x$. $\square$

Lemma 37. *Let X be a simplex and $\mu_1, \ldots, \mu_n$ probabilities on X with the same barycentre x. Let $A_1, \ldots, A_n$ be disjoint Borel subsets of X and $z_1, \ldots, z_n \in X$. Then*

$$\sum_{i=1}^{n} \mu_i(\{z_i\})\delta_{z_i}(A_i) \leqslant 1.$$

Proof. For each $i \in \{1, \ldots, n\}$ we have $\mu_i \geqslant \mu_i(\{z_i\})\varepsilon_{z_i}$. Hence, by Lemma 36(i) and (iv), we have

$$\delta_x = T\mu_i \geqslant T(\mu_i(\{z_i\})\varepsilon_{z_i}) = \mu_i(\{z_i\})\delta_{z_i}.$$

Therefore

$$\sum_{i=1}^{n} \mu_i(\{z_i\})\delta_{z_i}(A_i) \leqslant \sum_{i=1}^{n} \delta_x(A_i) = \delta_x\left(\bigcup_{i=1}^{n} A_i\right) \leqslant 1. \qquad \square$$

Lemma 38. *Assume that condition* (iii) *of Theorem* 1 *holds true. Then for every $x \in X \setminus \operatorname{ext} X$ there exists a G_δ-set separating x from* $\operatorname{ext} X$.

Proof. Let $x \in X \setminus \operatorname{ext} X$ be given. Since $\delta_x(\{x\}) = 0$ (see [1, p. 35, Remark 1]), we can select a compact set $K \subset X \setminus \{x\}$ with $\delta_x(K) \geqslant \frac{1}{2}$. Let f be a continuous function on X with values in $[0, 1]$, such that $f(x) = 0$ and $f = 1$ on K. By our assumption, H^f is Baire-one. Thus

$$G := \left\{ y \in X : f(y) = 0,\ \ H^f(y) \geqslant \tfrac{1}{2} \right\}$$

is a G_δ-set containing x and $G \cap \operatorname{ext} X = \emptyset$.

Indeed, if $y \in G \cap \operatorname{ext} X$, then $f(y) = 0$. On the other hand,

$$\tfrac{1}{2} \leqslant H^f(y) = \delta_y(f) = \varepsilon_y(f) = f(y),$$

a contradiction.

This observation finishes the proof. □

Remark 39. It follows from Lemma 38 that the perfectly unseparable compact set K from Definition 32 has no isolated points.

Lemma 40. *In the situation of* (*), *let x be a point of K and U be an open set intersecting K. Let $c > 0$. Then there are open sets V_1, V_2, such that $x \in V_1$, $V_2 \subset U$, $V_2 \cap K \neq \emptyset$ and $\mu(V_2) < c$ for every $y \in V_1$ and $\mu \in \mathcal{M}_y$.*

Proof. Suppose that the assertion does not hold. Then we have the following claim.

Claim 40.1. *For every $z \in U \cap K$ there exists a measure $\mu \in \mathcal{M}_x$, such that $\mu(\{z\}) \geqslant c$.*

Proof. Assume that the claim does not hold for some $z \in K \cap U$. By Lemma 19 and Theorem 2 the set

$$R_W := \{(y, \mu) \in X \times \mathcal{M}^1(X) : \mu \in \mathcal{M}_y, \mu(\overline{W}) \geqslant c\}$$

is closed for any neighbourhood W of z. Let π denotes the projection from $X \times \mathcal{M}^1(X)$ onto X. Since $R_{W_1} \cap R_{W_2} \supset R_{W_1 \cap W_2}$ for any couple W_1, W_2 of neighbourhoods of z, $\pi(\bigcap_W R_W) = \bigcap_W \pi(R_W)$. As

$$\bigcap_W R_W = \{(y, \mu) \in X \times \mathcal{M}^1(X) : \mu \in \mathcal{M}_y, \mu(\{z\}) \geqslant c\},$$

we have $x \notin \pi(\bigcap_W R_W)$. Thus, there exists an open set W containing z such that $x \notin \pi(R_W)$. Since $\pi(R_W)$ is compact, we can find an open set V containing x disjoint from $\pi(R_W)$. This concludes the proof because the open sets V and $W \cap U$ contradict our assumption. □

Fix $n \in \mathbb{N}$ such that $nc > 1$. Since K has no isolated points by Remark 39, we can use Lemma 34 to construct sequences $\{W_i(k)\}$, $i = 1, \ldots, n$, of nonempty open sets such that for $i = 1, \ldots, n$ and $k \in \mathbb{N}$

(i) $W_i(1)$, $i = 1, \ldots, n$, are pairwise disjoint;
(ii) $W_i(k) \subset U$;
(iii) $W_i(k) \cap K \neq \emptyset$;
(iv) $\overline{W_i(k+1)} \subset W_i(k)$;
(v) $\delta_y(W_i(k)) > 1 - \frac{1}{k}$ for each $y \in W_i(k+1) \cap K$.

Fix $i \in \{1, \dots, n\}$. Set $W_i := \bigcap_k W_i(k) = \bigcap_k \overline{W_i(k)}$ and pick a point

$$z_i \in W_i \cap K.$$

By (v) we have $\delta_{z_i}(W_i) = 1$. Indeed, $\delta_{z_i}(W_i(k)) > 1 - \frac{1}{k}$ for each $k \in \mathbb{N}$ and

$$\delta_{z_i}(W_i) = \delta_{z_i}\left(\bigcap_{k=1}^{\infty} W_i(k)\right) = \lim_{k\to\infty} \delta_{z_i}(W_i(k)).$$

According to Claim 40.1, there is a measure $\mu_i \in \mathcal{M}_x$, such that $\mu_i(\{z_i\}) \geqslant c$. Then it follows from Lemma 37 that

$$1 < nc \leqslant \sum_{i=1}^{n} \mu_i(\{z_i\}) = \sum_{i=1}^{n} \mu_i(\{z_i\})\delta_{z_i}(W_i) \leqslant 1,$$

a contradiction. $\square$

Now we are ready to prove the final lemma witnessing that (*) leads to a contradiction. Throughout the constructions in Lemma 41 and Section 8 we use the following notation. By $\{0,1\}^{<\mathbb{N}}$ we mean the set of all finite sequences of 0's and 1's. For a sequence $s \in \{0,1\}^{<\mathbb{N}}$ we write $|s|$ for the length of s. We adopt the convention that the length of the empty sequence $\emptyset$ is 0. If $s \in \{0,1\}^{<\mathbb{N}}$ and $i \in \{0,1\}$, we denote by $s^\wedge i$ the sequence $(s_1, \dots, s_{|s|}, i)$. If $\sigma \in \{0,1\}^{\mathbb{N}}$ is an infinite sequence and $n \in \mathbb{N}$, we write $\sigma\restriction n$ for the restriction $(\sigma_1, \dots, \sigma_n)$ of σ to the first n coordinates.

Lemma 41. *In the situation of* (*), *there exists a closed G_δ-set $C \subset K$, such that H^{χ_C} is not a Baire-one function.*

Proof. Let $\{\eta_n\}$ be a decreasing sequence of strictly positive numbers, such that $\sum_n \eta_n \leqslant \frac{1}{2}$. According to Lemma 38, we can assign to each point $x \in X \setminus \operatorname{ext} X$ a decreasing sequence of open sets $\{G(x,n)\}$, such that

$$x \in \bigcap_{n=1}^{\infty} G(x,n) \subset X \setminus \operatorname{ext} X.$$

We will construct points $x_s \in K$ and open sets U_s, V_s, $s \in \{0,1\}^{<\mathbb{N}}$, such that for each $s \in \{0,1\}^{<\mathbb{N}}$ the following conditions are fulfilled:

(a) $\overline{U}_{s^\wedge 0} \cup \overline{U}_{s^\wedge 1} \subset V_s$; $\overline{U}_{s^\wedge 0} \cap \overline{U}_{s^\wedge 1} = \emptyset$;
(b) both U_s and V_s intersect K;

(c) $x_s \in V_s$, $\overline{V}_s \subset U_s$;
(d) $x_{s^\wedge 0} = x_s$;
(e) $V_{s^\wedge 0} \subset G(x_{s^\wedge 0}, |s^\wedge 0|)$;
(f) if $y \in V_{s^\wedge 1} \cap K$, then $\delta_y(U_s) > 1 - 2^{-|s^\wedge 1|}$;
(g) if $y \in V_{s^\wedge 0}$, then

$$\delta_y\left(\bigcup\{V_{t^\wedge 1} : |t| = |s|\}\right) < \eta_{|s^\wedge 0|}.$$

To start the construction, set $U_\emptyset = V_\emptyset = X$ and pick $x_\emptyset \in K$ arbitrary. Then all the conditions are satisfied.

Suppose now that $n \in \mathbb{N} \cup \{0\}$ and that the objects have been constructed for every $s \in \{0,1\}^{<\mathbb{N}}$ with $|s| \leqslant n$. Find open sets U_t, $|t| = n+1$, intersecting K such that $x_s \in U_{s^\wedge 0}$, $|s| = n$, and (a) holds for them. Set $x_{s^\wedge 0} := x_s$, $|s| = n$. Using Lemma 34 find $V_{s^\wedge 1}$, $|s| = n$, intersecting K such that $\overline{V}_{s^\wedge 1} \subset U_{s^\wedge 1}$ and (f) is satisfied. For every $s \in \{0,1\}^{<\mathbb{N}}$ of length n we use Lemma 40 to find $V_{s^\wedge 0}$ and to shrink $V_{s^\wedge 1}$ in such a way that $V_{s^\wedge 1}$ still intersects K, $x_{s^\wedge 0} \in V_{s^\wedge 0}$, $\overline{V}_{s^\wedge 0} \subset U_{s^\wedge 0}$, $V_{s^\wedge 0} \subset G(x_{s^\wedge 0}, n+1)$ and (g) is satisfied. For any s with $|s| = n$ pick $x_{s^\wedge 1} \in K \cap V_{s^\wedge 1}$. This finishes the inductive step.

Set

$$C := \bigcap_{n=1}^{\infty} \bigcup\{U_s : |s| = n\}$$

and define a mapping $\varphi : C \to \{0,1\}^{\mathbb{N}}$ by $x \mapsto \sigma$ if $x \in \bigcap_n U_{\sigma\restriction n}$. For $\sigma \in \{0,1\}^{\mathbb{N}}$ set

$$U_\sigma := \bigcap_{n=1}^{\infty} U_{\sigma\restriction n} \quad \left(= \bigcap_{n=1}^{\infty} \overline{V}_{\sigma\restriction n} = \bigcap_{n=1}^{\infty} \overline{U}_{\sigma\restriction n}\right).$$

Then φ is a continuous mapping from C onto $\{0,1\}^{\mathbb{N}}$, even $\varphi(C \cap K) = \{0,1\}^{\mathbb{N}}$ since $U_\sigma \cap K \neq \emptyset$ for each $\sigma \in \{0,1\}^{\mathbb{N}}$.

Set

$$D_1 := \{\sigma \in \{0,1\}^{\mathbb{N}} : \sigma(n) = 0 \text{ for all but finitely many } n\text{'s}\},$$
$$D_2 := \{0,1\}^{\mathbb{N}} \setminus D_1.$$

Claim 41.1. *For any $\sigma \in D_2$ and $y \in U_\sigma \cap K$ it holds $\delta_y(U_\sigma) = 1$ (and hence $\delta_y(C) = 1$).*

Proof. Let W be any open set containing U_σ and $k \in \mathbb{N}$. By compactness there exists $n_1 \in \mathbb{N}$ with $U_{\sigma\restriction n_1} \subset W$. Let $n_2 \geqslant \max\{n_1, k\}$ be such that $\sigma(n_2+1) = 1$ (we recall that $\sigma \in D_2$). Since $y \in V_{(\sigma\restriction n_2)^\wedge 1} \cap K$, by (f) we get

$$\delta_y(W) \geqslant \delta_y(U_{\sigma\restriction n_2}) > 1 - 2^{-(n_2+1)} \geqslant 1 - 2^{-k}.$$

As k is arbitrary, $\delta_y(W) = 1$. By regularity of δ_y this finishes the proof. □

Claim 41.2. *For any $\sigma \in D_1$ and any $y \in U_\sigma$ it holds $\delta_y(C) \leqslant \frac{1}{2}$.*

Proof. Let $\sigma \in D_1$ and $y \in U_\sigma$ be given. According to (e), U_τ is a G_δ-set disjoint from $\operatorname{ext} X$ for each $\tau \in D_1$, and so $\delta_y(U_\tau) = 0$ for each $\tau \in D_1$ (see [1, Corollary I.4.12 and the subsequent Remark]). As D_1 is countable, we have

$$\delta_y\left(\bigcup\{U_\tau : \tau \in D_1\}\right) = \sum_{\tau\in D_1} \delta_y(U_\tau) = 0.$$

Choose $k \in \mathbb{N}$ such that $\sigma(n) = 0$ for $n \geqslant k$. By property (g) (using again the fact that D_1 is countable and hence $\bigcup\{U_\tau : \tau \in D_2\}$ is a G_δ, and thus a measurable set in C),

$$\delta_y\left(\bigcup\{U_\tau : \tau \in D_2\}\right) \leqslant \sum_{n=k}^{\infty} \delta_y\left(\bigcup\{U_{t^\wedge 1} : |t| = n\}\right) \leqslant \sum_{n=k+1}^{\infty} \eta_n \leqslant \frac{1}{2}.$$

Putting these two facts together we get $\delta_y(C) \leqslant \frac{1}{2}$. □

Remark 41.3. It is easy to check that we can even get $\delta_y(C) = 0$ in the previous claim. However, for our purposes the upper bound $\frac{1}{2}$ is sufficient.

We are now ready to conclude the proof of Lemma 41. Assuming that $f = H^{\chi_C}$ is a Baire-one function, f is Baire-one on $C \cap K$ as well. By Lemma 12 applied to $\varphi : C \cap K \to \{0,1\}^{\mathbb{N}}$ we get a mapping $\psi : \{0,1\}^{\mathbb{N}} \to C \cap K$ such that $f \circ \psi$ is a Baire-one function. Claims 41.1 and 41.2 imply that $(f \circ \psi)(\tau) = 1$ if $\tau \in D_2$ and $(f \circ \psi)(\tau) \leqslant \frac{1}{2}$ if $\tau \in D_1$. As D_1 and D_2 are both dense in $\{0,1\}^{\mathbb{N}}$, $f \circ \psi$ has no point of continuity on $\{0,1\}^{\mathbb{N}}$. As this is impossible for a Baire-one function (see Proposition 5(v)), the proof is finished. □

8. Condition (iii) implies that ext X is an H-set

In this section we prove the following proposition.

Proposition 42. *If condition* (iii) *of Theorem* 1 *holds*, ext X *is an H-set.*

Proof. Suppose that (iii) holds and $\operatorname{ext} X$ is not an H-set. We will show that this leads to a contradiction.

By the definition of an H-set we may fix a nonempty closed set $F \subset X$ such that

$$\overline{F \cap \operatorname{ext} X} = \overline{F \setminus \operatorname{ext} X} = F. \tag{7}$$

We establish the following auxiliary lemmas.

Lemma 42.1. *Let x be a point in $X \setminus \operatorname{ext} X$, U an open set intersecting F and $\eta > 0$. Then there exists an open set V intersecting F, such that $V \subset U$ and $\mu(V) < \eta$ for every $\mu \in \mathcal{M}_x$.*

Proof. Let x and U be as in the premise. Due to condition (7), $U \cap F \cap \operatorname{ext} X$ is an infinite set. Thus, we can find a point $y \in U \cap F \cap \operatorname{ext} X$ such that $\delta_x(\{y\}) < \eta$.

We claim that a suitable open neighbourhood V of y satisfies our requirements. Assume that this is not the case. Then for each open V containing y there is a measure $\mu_V \in \mathcal{M}_x$, such that $\mu_V(V) \geqslant \eta$. By passing to a subnet if necessary we may assume that $\mu_V \to \mu$. Then μ represents x (see Lemma 19) and for each open W containing y Theorem 2 gives

$$\mu(\overline{W}) \geqslant \limsup_V \mu_V(\overline{W}) \geqslant \limsup_V \mu_V(V) \geqslant \eta.$$

According to Lemma 3, $\mu(\{y\}) \geqslant \eta$. It follows from Lemma 21 that

$$\eta \leqslant \mu(\{y\}) \leqslant \delta_x(\{y\}).$$

But this contradicts our choice of the point y. Hence, there exists an open set V containing y, such that $\mu(V) < \eta$ for each $\mu \in \mathcal{M}_x$. Obviously we may achieve that $V \subset U$. This finishes the proof. $\square$

Lemma 42.2. *Let $\eta > 0$, $K \subset X$ be finite and U an open set intersecting F. Then there exist open sets V_i, $i = 0, 1$, such that $K \subset V_0$, $V_1 \subset U$, $V_1 \cap F \neq \emptyset$ and $(\overline{V}_0, \overline{V}_1)$ is η-singular.*

Proof. Let $\{x_i : i = 1, \ldots, n\}$ be an enumeration of the set $K \setminus \operatorname{ext} X$. We use Lemma 42.1 to obtain an open set W_1 intersecting F, such that $W_1 \subset U$ and $\mu(W_1) < \eta$ for each $\mu \in \mathcal{M}_{x_1}$. Another application of the lemma yields the existence of an open set W_2 intersecting F, such that $W_2 \subset W_1$ and $\mu(W_2) < \eta$ for every $\mu \in \mathcal{M}_{x_2}$. By repeating this process we get an open set W intersecting F, such that $W \subset U$ and $\mu(W) < \eta$ for each $\mu \in \bigcup_{i=1}^n \mathcal{M}_{x_i}$.

Further we adjust W in such a way that $W \cap K = \emptyset$ and pick a point $x \in W \cap F \cap \operatorname{ext} X$. Then $(K, \{x\})$ is η-singular and thus we may use Lemma 20 to get open sets V_0, V_1, such that $K \subset V_0$, $x \in V_1$ and $(\overline{V}_0, \overline{V}_1)$ is η-singular. Obviously we may also demand that $V_1 \subset U$. $\square$

Lemma 42.3. *Let U_0, U_1 be open sets intersecting F and $\eta > 0$. Then there exist open sets V_0, V_1 intersecting F, such that $V_i \subset U_i$, $i = 0, 1$, and $(\overline{V}_0, \overline{V_1})$ is η-singular.*

Proof. We choose a point $x \in U_0 \cap F$ and apply Lemma 42.2 with $K := \{x\}$ and $U := U_1$. Obviously we may achieve that $V_0 \subset U_0$. $\square$

Now we are going to construct a closed G_δ-set C, such that H^{χ_C} is not Baire-one.

Let $\{\eta_n\}$ be a decreasing sequence of strictly positive numbers tending to 0. As (iii) of Theorem 1 holds, we can, due to Lemma 38 assign to each point $x \in X \setminus \operatorname{ext} X$ a decreasing sequence of open sets $\{G(x, n)\}$, such that

$$x \in \bigcap_{n=1}^{\infty} G(x, n) \subset X \setminus \operatorname{ext} X.$$

If $s \in \{0, 1\}^{<\mathbb{N}}$ is a finite sequence of 0's and 1's, we denote by $z(s)$ the position where the digit 1 last occurs. If there is no 1 in s, we set $z(s) = 1$. We will construct points $x_s \in F$ and open sets V_s, W_s, $s \in \{0, 1\}^{<\mathbb{N}} \setminus \{\emptyset\}$, such that

(a) $x_0 \in F \setminus \operatorname{ext} X$, $x_1 \in F \cap \operatorname{ext} X$;
(b) $x_s \in W_s$, $x_{s^\wedge 0} = x_s$;
(c) $\overline{V}_{s^\wedge 0} \cap \overline{V}_{s^\wedge 1} = \emptyset$, $\overline{V}_{s^\wedge 0} \cup \overline{V}_{s^\wedge 1} \subset W_s \subset \overline{W}_s \subset V_s$;
(d) $x_{s^\wedge 1} \in F \cap \operatorname{ext} X$ if and only if $x_s \in F \setminus \operatorname{ext} X$;
(e) if $x_s \in F \cap \operatorname{ext} X$, then $\mu(V_s) > 1 - \eta_{|s|}$ for every $y \in W_s$ and $\mu \in \mathcal{M}_y$;
(f) if $x_s \in F \setminus \operatorname{ext} X$, then $W_s \subset G(x_s, |s|)$;
(g) for each $y \in V_s$ and $\mu \in \mathcal{M}_y$ holds

$$\mu\left(\bigcup \left\{\overline{V}_t : |t| = |s|, t \neq s\right\}\right) \leqslant \eta_{z(s)} \sum_{k=1}^{|s|} \frac{1}{2^k}.$$

To start the construction, we find a couple of open sets V_0 and V_1 intersecting F, such that $\overline{V}_0 \cap \overline{V}_1 = \emptyset$. We pick a point $x_0 \in (F \setminus \operatorname{ext} X) \cap V_0$. Using Lemma 42.2 we shrink V_0 and V_1 such that $x_0 \in V_0$, V_1 still intersects F and $(\overline{V}_0, \overline{V}_1)$ is $\frac{\eta_1}{2}$-singular. We select any point $x_1 \in V_1 \cap F \cap \operatorname{ext} X$ and find an open neighbourhood W_1 of x_1, such that $W_1 \subset \overline{W}_1 \subset V_1$ and $\mu(V_1) > 1 - \eta_1$ for every $y \in W_1$ and $\mu \in \mathcal{M}_y$ (here we use Lemma 23). We finish the first step of the construction by finding an open set W_0 with $x_0 \in W_0 \subset \overline{W}_0 \subset G(x_0, 1)$.

Assume now that the construction has been completed up to the nth stage. To begin with the construction of the objects of the $n+1$ stage, we find open sets V_t, $|t| = n+1$, that intersect F, satisfy (c) and $x_s \in V_{s^\wedge 0}$ if $|s| = n$. Set $x_{s^\wedge 0} := x_s$, $|s| = n$,

$$A := \{x_s \in F \cap \operatorname{ext} X : |s| = n\} \quad \text{and} \quad B := \{x_s \in F \setminus \operatorname{ext} X : |s| = n\}.$$

Using Lemma 42.2 we can shrink the open sets V_t, $|t| = n+1$, such that

(h) $x_{s^\wedge 0} \in V_{s^\wedge 0}$;
(i) $V_{s^\wedge 1}$ intersects F if $|s| = n$;
(j) $(\bigcup\{\overline{V}_{s^\wedge 0} : |s| = n\}, \bigcup\{\overline{V}_{s^\wedge 1} : |s| = n\})$ is $\frac{1}{2}2^{-(n+1)}\eta_{n+1}$-singular.

Indeed, for each s of length n we use Lemma 42.2 to find an open set G_s containing $A \cup B$ and an open set U_s, such that $\emptyset \neq U_s \cap F$, $U_s \subset V_{s^\wedge 1}$ and $(\overline{G}_s, \overline{U}_s)$ is $2^{-2(n+1)}\eta_{n+1}$-singular. Then for each s of length n we can replace $V_{s^\wedge 0}$ with $V_{s^\wedge 0} \cap \bigcap\{G_t : |t| = n\}$ and $V_{s^\wedge 1}$ with U_s.

If $y \in \bigcup\{\overline{V}_{s^\wedge 0} : |s| = n\}$ and $\mu \in \mathcal{M}_y$, then

$$\mu\left(\bigcup\{\overline{V}_{s^\wedge 1} : |s| = n\}\right) = \sum_{|s|=n} \mu(\overline{V}_{s^\wedge 1}) < \sum_{|s|=n} 2^{-2(n+1)}\eta_{n+1} = \frac{1}{2}2^{-(n+1)}\eta_{n+1}.$$

Similarly, we can show that

$$\mu\left(\bigcup\{\overline{V}_{s^\wedge 0} : |s| = n\}\right) < \frac{1}{2}2^{-(n+1)}\eta_{n+1}$$

if $y \in \bigcup\{\overline{V}_{s^\wedge 1} : |s| = n\}$ and $\mu \in \mathcal{M}_y$. Thus condition (j) is fulfilled.

Now we want to achieve that for each $s \in \{0,1\}^{<\mathbb{N}}$ of length n

$$\left(\overline{V}_{s^\wedge 1}, \bigcup\{\overline{V}_{t^\wedge 1} : |s| = |t| = n,\, s \neq t\}\right) \text{ is } \frac{1}{2}2^{-(n+1)}\eta_{n+1} \text{ -singular.} \tag{8}$$

To this end, let $\{V_i\}_{i=1}^{2^n}$ be an enumeration of $\{V_{s^\wedge 1} : |s| = n\}$.

According to Lemma 42.3, we can shrink V_1 and V_2 in such a way that both sets intersects F and $(\overline{V}_1, \overline{V}_2)$ is $2^{-2(n+1)}\eta_{n+1}$-singular. Another adjustment of V_1 and V_3 ensures that $(\overline{V}_1, \overline{V}_3)$ is $2^{-2(n+1)}\eta_{n+1}$-singular. After finitely many steps we shrink our sets in such a way that $(\overline{V}_1, \overline{V}_i)$ is $2^{-2(n+1)}\eta_{n+1}$-singular for each $i = 2, \ldots, 2^n$. Thus $(\overline{V}_1, \bigcup\{\overline{V}_i : i = 2, \ldots, 2^n\})$ is $\frac{1}{2}2^{-(n+1)}\eta_{n+1}$-singular.

Then we apply the same procedure to the set V_2 and get adjustments of sets $\{V_i\}_{i=1}^{2^n}$, such that $(\overline{V}_2, \bigcup\{\overline{V}_i : i = 1, \ldots, 2^n, i \neq 2\})$ is $\frac{1}{2}2^{-(n+1)}\eta_{n+1}$-singular.

We go on with this procedure for every set V_i, $i = 1, \ldots, 2^n$ and finally we get that our sets $V_{s^\wedge 1}$, $|s| = n$, satisfy (8).

We claim that the sets $\{V_t : |t| = n+1\}$ satisfy condition (g). Indeed, if $t = s^\wedge 1$ for some s of length n, let $y \in \overline{V}_t$ and $\mu \in \mathcal{M}_y$ be given. Then condition (j) along with (8) gives

$$\begin{aligned}\mu\left(\bigcup\{\overline{V}_u : |u| = n+1, u \neq t\}\right) &= \mu\left(\bigcup\{\overline{V}_{u^\wedge 1} : |u| = n, u \neq s\}\right)\\ &\quad + \mu\left(\bigcup\{\overline{V}_{u^\wedge 0} : |u| = n\}\right)\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{2}2^{-(n+1)}\eta_{n+1} + \frac{1}{2}2^{-(n+1)}\eta_{n+1} \\
&= 2^{-(n+1)}\eta_{n+1}.
\end{aligned}
$$

If $t = s^\wedge 0$ for some s of length n, let $y \in \overline{V}_t$ and $\mu \in \mathcal{M}_y$ be given. Then the inductive assumption along with condition (j) gives

$$
\begin{aligned}
\mu\left(\bigcup\{\overline{V}_u : |u| = n+1, u \neq t\}\right) &\leqslant \mu\left(\bigcup\{\overline{V}_v : |v| = n, v \neq s\}\right) + \mu\left(\overline{V}_{s^\wedge 1}\right) \\
&< \eta_{z(s)} \sum_{k=1}^{n} 2^{-k} + \frac{1}{2}2^{-(n+1)}\eta_{n+1} \\
&< \eta_{z(s)} \sum_{k=1}^{n+1} 2^{-k}.
\end{aligned}
$$

In both cases we have verified that the family $\{V_t : |t| = n+1\}$ satisfies condition (g).

Now we choose points $x_{s^\wedge 1} \in V_{s^\wedge 1} \cap F$ such that $x_{s^\wedge 1} \in \operatorname{ext} X$ if $x_s \notin \operatorname{ext} X$ and $x_{s^\wedge 1} \notin \operatorname{ext} X$ if $x_s \in \operatorname{ext} X$. Further we pick open sets W_t, $|t| = n+1$, such that $x_t \in W_t \subset \overline{W}_t \subset V_t$ and either W_t satisfies (e) if $x_t \in F \cap \operatorname{ext} X$ (here we use Lemma 23) or W_t satisfies (f) if $x_t \in F \setminus \operatorname{ext} X$. This finishes the inductive step of the construction.

We define

$$
C := \bigcap_{n=1}^{\infty} \bigcup_{|s|=n} V_s
$$

and $\varphi : C \to \{0,1\}^{\mathbb{N}}$ by

$$
x \mapsto \sigma \in \{0,1\}^{\mathbb{N}} \quad \text{if and only if} \quad x \in \bigcap_{n=1}^{\infty} V_{\sigma\restriction n}.
$$

By condition (c), C is a closed G_δ-set in X and φ is a continuous mapping of C onto the Cantor set $D = \{0,1\}^{\mathbb{N}}$. We set

$$
A := \{x_s : s \in \{0,1\}^{<\mathbb{N}}\} \cap \operatorname{ext} X \quad \text{and} \quad B := \{x_s : s \in \{0,1\}^{<\mathbb{N}}\} \setminus \operatorname{ext} X.
$$

Then it follows from (d) that both the sets $\varphi(A)$ and $\varphi(B)$ are dense in D.

Given $s \in \{0,1\}^{<\mathbb{N}}$, let $\sigma(s)$ denote the sequence $\{s_1, \ldots, s_{|s|}, 0, 0, \ldots\}$. For $\sigma \in D$ we write V_σ for the set $\bigcap_n V_{\sigma\restriction n}$. We will need the following lemma.

Lemma 42.4. *Let $x_s \in A$ and y be a point of $V_{\sigma(s)}$. Then $\delta_y(C) = 1$. Similarly, if $x_s \in B$ and $y \in V_{\sigma(s)}$, then $\delta_y(C) \leqslant \eta_{z(s)}$.*

Proof. Let x_s be in A and y in $V_{\sigma(s)}$. Let U be any open set containing $V_{\sigma(s)}$. Since

$$V_{\sigma(s)} = \bigcap_{n=1}^{\infty} \overline{V}_{\sigma(s)\restriction n} = \bigcap_{n=1}^{\infty} \overline{W}_{\sigma(s)\restriction n}$$

by condition (c), there exists an $n \in \mathbb{N}$, such that $V_{\sigma(s)\restriction n} \subset U$. According to (e),

$$\delta_y(U) > 1 - \eta_k$$

for every $k \geqslant n$. Thus $\delta_y(U) = 1$ for every open set containing $V_{\sigma(s)}$. Hence $\delta_y(V_{\sigma(s)}) = 1$.

Concerning the second assertion, let $x_s \in B$ and $y \in V_{\sigma(s)}$ be given. Let $K \subset C$ be a compact set disjoint from $V_{\sigma(s)}$. Due to the compactness of K there exists $n \in \mathbb{N}$ with $n \geqslant |s|$ such that

$$K \subset \bigcup \{V_t : |t| = n, t \neq (s_1, \ldots, s_{|s|}, 0, \ldots, 0)\}.$$

It follows from condition (g) that

$$\delta_y(K) \leqslant \eta_{z(s)} \sum_{k=1}^{n} \frac{1}{2^k} \leqslant \eta_{z(s)}.$$

The regularity of δ_y implies that

$$\delta_y(C \setminus V_{\sigma(s)}) \leqslant \eta_{z(s)}.$$

Since $\delta_y(V_{\sigma(s)}) = 0$, (by (f) it is a G_δ-set disjoint from $\operatorname{ext} X$), $\delta_y(C) \leqslant \eta_{z(s)}$ which is the sought conclusion. $\square$

We define

$$f(x) := H^{\chi_C}(x), \quad x \in X.$$

It follows from the previous lemma that $f(x) = 1$ for every $x \in V_{\sigma(s)}$ if $x_s \in A$. Also we get that $f(x) \leqslant \eta_{z(s)}$ for every $x \in V_{\sigma(s)}$ if $x_s \in B$.

We are now ready to conclude the reasoning. Assuming that f is a Baire-one function, we employ Lemma 12 and get the corresponding mapping $\psi : D \to C$ such that $f \circ \psi$ is a Baire-one function. Lemma 42.4 implies that $(f \circ \psi)(\varphi(x_s)) = 1$ if $x_s \in A$ and

$(f \circ \psi)(\varphi(x_s)) \leqslant \eta_{z(s)}$ if $x_s \in B$. Thus $f \circ \psi$ has no point of continuity on C contradicting the fact that it is a Baire-one function (see Proposition 5(v)). □

9. Open questions

Although the abstract Dirichlet problem for Baire-one functions is completely solved now, some related questions remain open. Let us state some of them.

Question 1. Are the following assertions equivalent?

(i) X is a simplex and $\operatorname{ext} X$ is Lindelöf.
(ii) Any bounded continuous function on $\operatorname{ext} X$ can be extended to an affine Baire-one function.

The implication (i) $\Longrightarrow$ (ii) does hold by Jellett [9, Theorem]. The converse holds within the class of Stacey simplices due to [10, Theorem 2]. However, it is not known whether (ii) $\Longrightarrow$ (i) holds in general.

Question 2. Let $\operatorname{ext} X$ be Lindelöf. Is $\operatorname{ext} X$ necessarily hereditarily Baire?

If X is any compact convex set, $\operatorname{ext} X$ is necessarily Baire (in fact, α-favourable, see [5, Theorem 27.9]). However, $\operatorname{ext} X$ need not be hereditarily Baire, since any completely regular space is homeomorphic to a closed subset of $\operatorname{ext} X$ for a simplex X, see [19, Corollary 2]. On the other hand, if X is metrizable, then $\operatorname{ext} X$ is G_δ and hence hereditarily Baire. Moreover, if X is a Stacey simplex with $\operatorname{ext} X$ Lindelöf, then $\operatorname{ext} X$ is hereditarily Baire by Kalenda [10, Theorem 2]. If X is a compact convex set such that $\operatorname{ext} X$ is $\mathcal{K}$-countably determined, $\operatorname{ext} X$ is hereditarily Baire as follows from [20, Theórème 2].

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